ON THE ONE-DIMENSIONAL TRANSLATION GROUP AND SEMI-GROUP IN CERTAIN FUNCTION SPACES

BY

BERTIL NYMAN

UPPSALA 1950
APPELBERGS BOKTRYCKERI AB

ERRATA

Page:	Line:	For:	Read:
8	28	assures	assure
24	17	follows	follow
29	25, 26	if and . . ., $A > 0$,	if and only if, in any interval $(0, A)$, $A > 0$, at least one function is not $= 0$ a.e.,
44	10	$G(it)$	$G_0(it)$
50	20, 21	and the . . . section 10.	as the least linear, closed manifold, containing A and invariant under translation to the left. $\tilde{\Lambda}_A$ is then defined in the same way as Λ_A.
51	7	concider	consider
51	10	$l^\infty(-\infty, \infty)$	$l^\infty(0, \infty)$

ON THE ONE-DIMENSIONAL TRANSLATION GROUP AND SEMI-GROUP IN CERTAIN FUNCTION SPACES

Inaugural Dissertation

by

BERTIL NYMAN
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BY DUE PERMISSION OF THE PHILOSOPHICAL
FACULTY OF THE UNIVERSITY OF UPPSALA,
TO BE PUBLICLY DISCUSSED AT THE
INSTITUTE OF MATHEMATICS,
MAY 25TH, 1950, AT 10 A. M.,
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UPPSALA 1950
APPELBERGS BOKTRYCKERI AB

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2007/30.

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The subject of this dissertation was suggested to me by my teacher, Professor Arne Beurling. During my years as his student, Professor Beurling has given me most generously of his always valuable advice and outstanding instruction, and he has shown me the greatest kindness and encouragement. I very much appreciate this opportunity for expressing to him my sincere gratitude.

I further wish to thank Mr. John Wermer for his kind help with the revision of my English text.

Uppsala, May 1950.

Bertil Nyman.

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INTRODUCTION

1. The problem. — A general problem concerning translations of functions is, briefly, the following: Let B be a linear space of complex-valued functions on a locally compact Abelian topological group G, and let C be a non-empty linear subset of B, invariant under translation, and closed in a certain topology in B. Under what conditions is it true that C always contains a character of G?

The first theorem in this field is due to BEURLING [2]. He has proved the following: Let B be the space of bounded, uniformly continuous functions on the real line. Let $\varphi(x) \not\equiv 0$ be a function in B, and let C_φ be the closure of the linear manifold spanned by the set $\varphi(x-\tau)$, $-\infty < \tau < \infty$, in the narrow topology in B (see [2]). Then C_φ contains at least one pure oscillation e^{-itx}. — The points it, such that $e^{-itx} \in C_\varphi$, form a closed set on the imaginary axis in an s-plane, $s = \sigma + it$. This set is termed the spectral set of $\varphi(x)$.

GODEMENT [10] has proved an analogous theorem in the space $L^\infty(G)$, i.e. the space of bounded functions on a locally compact Abelian topological group G; the theorem states that every non-empty weakly closed linear subset of $L^\infty(G)$, which is invariant under translation, contains at least one character of G. SCHWARTZ [13] has proved another theorem of this kind, in the space of continuous functions on the real line in a certain topology. Furthermore, BEURLING [7] has established such a theorem, an "extinction theorem", in a certain Hilbert space.

2. The spaces of functions considered. — In this paper we shall study the above problem in a space L^∞_p, associated to a weight-function $p(x)$ of the kind introduced by BEURLING [1]. (For the definition of a weight-function $p(x)$ and its associated spaces L^1_p and L^∞_p, see also page 35 of the present paper.)

In L_p^∞ we shall use the weak topology, the weak closure of a subset C of L_p^∞ being defined in the following way: A function $\varphi(x)$ in L_p^∞ belongs to the weak closure of C if and only if the relation

$$\int_{-\infty}^{\infty} k(x)\,\psi(x)\,dx = 0, \text{ for all } \psi \in C,$$

where $k(x) \in L_p^1$, implies that

$$\int_{-\infty}^{\infty} k(x)\,\varphi(x)\,dx = 0.$$

The spectral set of a function $\varphi(x)$ in L_p^∞ can be defined in a way similar to that above, as the set of all complex numbers s, such that e^{-sx} belongs to the weak closure of the span of translations of $\varphi(x)$. The spectral set thus defined will be a closed set contained in a strip $-\alpha \leq \sigma \leq \beta$, α and $\beta \geq 0$. In certain cases the strip can be a half-plane, or reduce to the imaginary axis.

In a lecture, Beurling has proved that his theorem in [2] can be extended to the space L_p^∞, when $p(x)$ is of non-quasianalytic type. (For the definition of this term, see page 36.) Furthermore, he has proved that the two spectral sets, defined respectively by aid of the weak, and by aid of the (stronger) narrow topology, coincide.

In this paper we shall show that the theorem in question cannot in general be extended to spaces L_p^∞, where $p(x)$ is of analytic or quasianalytic type. In these spaces, the above defined notion of spectral set will therefore, in general, lose its interest. However, the following "closure problem", which is intimately connected with the extinction problem in L_p^∞, remains of considerable interest: What conditions on a given function $k(x)$ in L_p^1 assures that the set of translations of $k(x)$ is fundamental on L_p^1?

3. Summary of results obtained and methods used. — After having proved some simple preliminary lemmas in chapter 1, we shall study in chapter 2 the space $L^\infty(0, \infty)$, which may be

regarded as the space L_p^∞, where $p(x)$ is the weight-function of analytic type, which is $=1$ for $x \geq 0$, and $=+\infty$ for $x<0$. In this case we are obviously concerned with translations in only one direction. We shall prove that the extinction theorem is valid in the sense that if a function has an empty spectral set, then some translation to the left of it must be $=0$ a.e. — In the last sections of chapter 2, we shall briefly touch the corresponding problem in $L^2(0, \infty)$.

In chapter 3 we shall first consider the closure problem in a space L_p^1, where $p(x)$ is a weight-function of analytic type of a certain special kind. In this case, the WIENER condition,

$$\int_{-\infty}^{\infty} k(x)\, e^{-sx}\, dx \not\equiv 0$$

in the relevant domain, turns out to be not sufficient for closure. In proving this, we shall construct functions $\varphi(x) \not\equiv 0$ in L_p^∞, which have empty spectral sets. — We shall then prove by an example that functions with the last mentioned properties also exist in L_p^∞ when $p(x)$ is of quasianalytic type, at least for some $p(x)$. — The author hopes to be able to study the questions in this chapter in greater generality later on.

Finally, in chapter 4, we shall consider the extinction problem in $l^\infty(0, \infty)$, i.e. the space of bounded sequences $\{a_n\}_0^\infty$. This case is closely related to that in chapter 2.

For the treatment of problems of the kind we have been discussing, various different methods have been developed. Without making any pretentions to completeness, we mention, to begin with, the method devised by BEURLING in [2]. Secondly, methods pertaining to the theory of integral equations and Fourier integrals have been used (WIENER [14], CARLEMAN [8], BEURLING [3, 4, 5, 6]). Finally, methods from the general theory of normed rings are of great importance (BEURLING [1], GELFAND [9], GODEMENT [10]). — In chapter 2 and 3 of the present paper we shall avail ourselves of the second of the above-mentioned methods; in chapter 4 we shall use a method related to the first of them.

CHAPTER 1

SOME LEMMAS CONCERNING ANALYTIC CONTINUATION AND FUNCTIONS HOLOMORPHIC IN A HALF-PLANE

4. A lemma concerning analytic continuation. — In order not to disturb the argument in what follows, we shall collect in this chapter some simple lemmas, mainly well known, concerning analytic continuation and functions holomorphic in a half-plane.

Lemma 1. *Let $u(z)$ be a function harmonic in a half-circle C, defined by $|z|<R$, $x>0$, and suppose that*

$$|u(z)|<M \text{ in } C, \tag{1}$$

$$\lim_{x\to+0} u(x, y)=0 \text{ a.e., } |y|<R. \tag{2}$$

Then

$$\lim_{x\to+0} u(x, y)=0$$

uniformly for $|y|\leq R'<R$.

Proof: Choose δ and ϱ such that $\sqrt{R'^2+\delta^2}<\varrho<R$. Let D_k be the half-circle $\left|z-\frac{1}{k}\right|<\varrho$, $x>\frac{1}{k}$, Γ_k its boundary and $G_k(\omega, z)$ Green's function for D_k. For $k=\infty$ we use the notations D, Γ

and $G(\omega, z)$. Let z be a point in the rectangle $0 < x \leq \delta$, $|y| \leq R'$. For all sufficiently large k, it holds that $z \in D_k \subset C$ and that $\frac{\partial G_k(\omega, z)}{\partial n}$ is uniformly bounded on Γ_k. If we represent $u(z)$ by Green's formula in D_k, we may thus, by (1) and (2), let $k \to \infty$, obtaining

$$u(z) = \frac{1}{2\pi} \int_{\Gamma} \frac{\partial G(\omega, z)}{\partial n} u(\omega)\, ds,$$

and the lemma follows immediately from this representation.

Consider now a function $f(z) = u + iv$, which is holomorphic in a domain D situated in the half-plane $x > 0$, and suppose that the segment (ia, ib) on the imaginary axis is a part of the boundary of D (no inner point of the segment being a limit point of boundary points not belonging to the segment). Suppose further that $|v(x, y)| < M$ in D and that $\lim_{x \to +0} v(x, y) = 0$ a.e., $a < y < b$. It follows from the lemma that this last relation holds uniformly on every interval $a + \varepsilon < y < b - \varepsilon$, $\varepsilon > 0$. As is well known, this implies that $f(z)$ can be analytically continued over the segment (ia, ib), and is holomorphic in the domain $D + D'$, where D' is the image of D by reflection in the imaginary axis.

This easily yields the following lemma:

Lemma 2. *Let $f_1(z)$ be holomorphic in D, and $f_2(z)$ holomorphic in D', and suppose that*

$$|f_1(x + iy) - f_2(-x + iy)| < M, \quad z \in D,$$

$$\lim_{x \to +0} (f_1(x + iy) - f_2(-x + iy)) = 0 \ \text{a.e.}, \quad a < y < b.$$

Then $f_1(z)$ and $f_2(z)$ are analytic continuations of each other.

For the proof we need only observe that the functions $f_1(z) + \overline{f_2(-\bar{z})}$ and $i(f_1(z) - \overline{f_2(-\bar{z})})$ are holomorphic in D, and both have imaginary parts subject to the conditions of Lemma 1. Consequently, they can be continued to D', and the lemma

follows. — For other theorems of this kind we refer to CARLEMAN ([8], page 38).

5. **A class S of functions holomorphic in a half-plane.** — Let us now consider the class S of functions $F(s)$, which are holomorphic in the half-plane $\sigma > 0$, and are such that

$$\int_{-\infty}^{\infty} \frac{|\log|F(iy)||}{1+y^2}\, dy < \infty.$$

The functions in S shall furthermore have the property that they can be uniquely factorized in the form

$$F(s) = F_0(s)\, F_1(s), \tag{3}$$

where

$$F_0(s) = e^{-As}\, B(s)\, \exp\left\{ -\int_{-\infty}^{\infty} \frac{1 - iys}{s - iy}\, d\mu(y) \right\}, \tag{4}$$

$$F_1(s) = c\, \exp\left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - iys}{s - iy}\, \frac{\log|F(iy)|}{1+y^2}\, dy \right\}. \tag{5}$$

$B(s)$ is the Blaschke product containing the zeros $\{a_n\}_1^\infty$ of $F(s)$ in the half-plane $\sigma > 0$, and

$$\sum_{n=1}^{\infty} \frac{\mathrm{Re}\,(a_n)}{1+|a_n|^2} < \infty \tag{6}$$

(each a_n being counted with its multiplicity). A is a real constant ≥ 0, c a complex constant of modulus 1, and $\mu(y)$ a real nondecreasing bounded function, such that its points of increase form a set of measure zero.

It is well known that $F(s) \in S$ if, for instance, $F(s)$ is bounded in $\sigma > 0$, or satisfies a Hardy condition

$$\int_{-\infty}^{\infty} |F(\sigma + it)|^p \, dt < M, \quad \sigma > 0,$$

for some $p > 0$. For the proof we refer to KRYLOFF [12] and KARHUNEN [11], in which papers also further references are to be found. In particular, if $F(s)$ is the Laplace transform of a function $f(x)$ in $L^1(0, \infty)$ or $L^2(0, \infty)$, then $F(s) \in S$.

In accordance with the terminology introduced by BEURLING [7], we shall call F_0 the "inner factor" and F_1 the "outer factor" of F, and shall say that F_0 is a divisor of the inner factor

$$G_0(s) = e^{-A_1 s} B_1(s) \exp\left\{ -\int_{-\infty}^{\infty} \frac{1 - iys}{s - iy} d\mu_1(y) \right\}$$

if G_0/F_0 is an inner factor, i.e. if $A_1 \geq A$ and the zeros of B form a subset of the zeros of B_1 and $\mu_1 - \mu$ is nondecreasing. More generally, the largest common factor of two inner factors F_0 and G_0 is defined as the inner factor which has the characteristics A_2, $B_2(s)$ and $\mu_2(y)$, defined in the following way: $A_2 = \min(A, A_1)$, and the set of zeros of B_2 is the intersection of the sets of zeros of B and B_1. Furthermore, μ_2 is the largest common minorant of μ and μ_1, considered as nonnegative completely additive set functions.

6. An elementary lemma concerning functions in S.

Lemma 3. *If $F(s) \in S$, then, as $R \to \infty$,*

$$\int_{|s|<R, \sigma>0} \int |\log|F(s)|| \, d\sigma \, dt = \frac{2A}{3} R^3 + o(R^3). \tag{7}$$

Proof: Let us write

$$F(s) = e^{-As} B(s) F_2(s). \tag{8}$$

By (3), (4) and (5), we have

$$|\log|F_2(s)|| = \left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma\log|F(iy)|}{\sigma^2+(y-t)^2}\,dy - \int_{-\infty}^{\infty}\frac{\sigma(1+y^2)}{\sigma^2+(y-t)^2}\,d\mu(y)\right| \leq$$

$$\leq \int_{-\infty}^{\infty}\frac{\sigma(1+y^2)}{\sigma^2+(y-t)^2}\,d\alpha(y),$$

where $\alpha(y)$ is a nondecreasing bounded function. Using this inequality, an elementary calculation shows that the integral in (7) with $F(s)$ replaced by $F_2(s)$ is less than

$$\int_{-\infty}^{\infty}\frac{1+y^2}{y}\left\{\frac{R^2-y^2}{2}\log\left|\frac{R+y}{R-y}\right| + R\,y\right\}d\alpha(y),$$

which is $o(R^3)$ as $R\to\infty$.

We now turn to the Blaschke product. Let $s = re^{i\varphi}$, and $a_n = \varrho_n e^{i\Theta_n}$, $n = 1, 2, \ldots$, be the zeros of $F(s)$ in $\sigma > 0$. We must show that

$$(9)\qquad \iint_{|s|<R,\,\sigma>0}\log\left|\frac{1}{B(s)}\right|d\sigma\,dt = \sum_{n=1}^{\infty}\int_0^R r\,dr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log\left|\frac{re^{i\varphi}+\bar{a}_n}{re^{i\varphi}-a_n}\right|d\varphi = o(R^3).$$

By derivation with respect to r it is seen that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log\left|\frac{re^{i\varphi}+\varrho\,e^{-i\Theta}}{re^{i\varphi}-\varrho\,e^{i\Theta}}\right|d\varphi = \int_0^q g(x,\Theta)\,dx = G(r,\varrho,\Theta),$$

where $g(x,\Theta) = \dfrac{2}{x}\operatorname{arccot}\dfrac{1-x^2}{2x\cos\Theta}$ and $q = \min\left(\dfrac{r}{\varrho},\dfrac{\varrho}{r}\right)$.

Let us assume that $R>1$. Using the inequality $g(x,\Theta) < 4\cos\Theta/(1-x)$, $0<x<1$, we obtain, after change of variable,

$$\int_0^R r\,G(r,\varrho,\Theta)\,dr < 4\varrho^2\cos\Theta\int_0^{\frac{R}{\varrho}} x\,\log\frac{1}{1-\min\left(x,\frac{1}{x}\right)}\,dx.$$

For $\varrho \geq R$, this is $< \frac{MR^3}{2\varrho}\cos\Theta < MR^3\frac{\varrho\cos\Theta}{1+\varrho^2}$. For $\varrho < R$, the right member is $< \frac{MR\varrho}{2}\cos\Theta < MR^3\frac{\varrho\cos\Theta}{1+\varrho^2}$, or, for $\varrho \leq \sqrt{R}$, $< MR^2\frac{\varrho\cos\Theta}{1+\varrho^2}$.

Thus we obtain

$$\iint_{|s|<R,\,\sigma>0}\log\left|\frac{1}{B(s)}\right|d\sigma\,dt < MR^3\sum_{\varrho_n>\sqrt{R}}\frac{\mathrm{Re}\,(a_n)}{1+\varrho_n^2} + MR^2\sum_{\varrho_n\leq\sqrt{R}}\frac{\mathrm{Re}\,(a_n)}{1+\varrho_n^2},$$

from which (9) follows, the series (6) being convergent.

It is obvious that the factor e^{-As} in (8) gives rise to the term $\frac{2A}{3}R^3$ in (7), and thus the lemma is proved.

7. A lemma concerning Laplace transforms of functions in $L^1(0,\infty)$ or $L^2(0,\infty)$.

Lemma 4. *Let $f(x)$ be a function in $L^1(0,\infty)$ or $L^2(0,\infty)$ and suppose that $f(x)=0$ a.e. in $(0,a)$, $a\geq 0$, but not $=0$ a.e. in any interval $(a,a+\varepsilon)$, $\varepsilon>0$. Let $F(s)$ be the Laplace transform of $f(x)$ and A the constant which appears in (4). Then $A=a$.*

Proof: We have

$$F(s)=\int_0^\infty f(x)\,e^{-sx}\,dx,$$

where we first assume that $f(x)\in L^2(0,\infty)$. By the factorization of $F(s)$, we obtain

$$\log |F(s)|^2 \leq -2A\sigma + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma \log |F(iy)|^2}{\sigma^2 + (y-t)^2}\, dy \leq$$

$$\leq -2A\sigma + \log\left\{ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma |F(iy)|^2}{\sigma^2 + (y-t)^2}\, dy \right\},$$

which yields

$$\int_{-\infty}^{\infty} |F(\sigma + it)|^2\, dt \leq e^{-2A\sigma} \int_{-\infty}^{\infty} |F(iy)|^2\, dy, \quad \sigma \geq 0.$$

From this it follows in a well known manner that $f(x) = 0$ a.e. in $(0, A)$, i.e. $A \leq a$.

In order to get an inequality in the opposite direction, we observe that $|F(s)| \leq \frac{M}{\sqrt{\sigma}} e^{-a\sigma}$. Thus

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |F(re^{i\varphi})|\, d\varphi \leq -2a.$$

However, from the foregoing lemma it easily results that this must be $\geq -2A$, hence $A \geq a$, or $A = a$.

If $f(x) \in L^1(0, \infty)$ we form the function $g(x) = e^{-x} \int_0^x f(y)\, e^y\, dy$,

which $\in L^2(0, \infty)$ and has the Laplace transform $F(s)/(1+s)$. In the factorization of this last function, the constant A remains unchanged, and hence $(0, A)$ is the maximal interval from the origin in which $g(x) = 0$, from which it follows that $(0, A)$ is the maximal interval in which $f(x) = 0$ a.e.

CHAPTER 2

ON TRANSLATIONS OF FUNCTIONS IN $L^p(0, \infty)$, $p = 1, 2, \infty$

8. Definitions. — Let $\varphi(x)$ be a function in $L^\infty(0, \infty)$. For the sake of convenience, we define $\varphi(x) \equiv 0$ for $x < 0$. Let C_φ be the closure of the linear manifold spanned by the set

$$\varphi(x+\tau), \ \tau \geq 0,$$

in the weak topology of bounded functions over $(0, \infty)$ (see page 8).

The spectral set of $\varphi(x)$, Λ_φ, is defined as the set of all complex numbers s, such that $e^{-sx} \in C_\varphi$. Λ_φ is a closed set situated in the half-plane $\sigma \geq 0$. If $x^\nu e^{-sx} \in C_\varphi$ for $\nu = 0, 1, \ldots, q-1$, while $x^q e^{-sx}$ does not $\in C_\varphi$, we shall say that s is a point of Λ_φ of the multiplicity q. It is obvious that a point of the spectral set, which lies on the imaginary axis, must have the multiplicity 1.

If $C_\varphi \equiv L^\infty(0, \infty)$, Λ_φ consists of the entire half-plane $\sigma \geq 0$, each point in $\sigma > 0$ having the multiplicity ∞. If C_φ is a proper subset of $L^\infty(0, \infty)$, we shall say that $\varphi(x)$ is "mean-periodic", in accordance with the terminology introduced by Schwartz [13].

9. The extinction theorem in $L^\infty(0, \infty)$.

Theorem 1. *If no translation $\varphi(x+\tau)$, $\tau \geq 0$, is $= 0$ a.e., then Λ_φ is not empty.*

Proof: This assertion can be restated as follows: Under the above condition, every $k(x)$ in $L^1(0,\infty)$ which satisfies an integral equation of the form

$$\int_0^\infty k(x)\,\varphi(x+y)\,dx = \psi(y), \tag{10}$$

where $\psi(y) \equiv 0$ for $y \geq 0$, must also satisfy the equation

$$\int_0^\infty k(x)\,e^{-sx}\,dx = 0,$$

for at least one fixed s, not depending on $k(x)$, but only on $\varphi(x)$.

Let us assume that $\varphi(x)$ is mean-periodic. Then $L^1(0,\infty)$ must contain a function $k(x) \not\equiv 0$, which satisfies an equation of the form (10). Let us define $k(x) \equiv 0$ for $x < 0$, and set

$$K(s) = \int_0^\infty k(x)\,e^{-sx}\,dx.$$

$K(s)$ is holomorphic in $\sigma > 0$ and continuous in the closed half-plane $\sigma \geq 0$. We further introduce the transforms

$$\Phi(s) = \int_0^\infty \varphi(x)\,e^{sx}\,dx,$$

$$\Psi(s) = \int_{-\infty}^0 \psi(x)\,e^{sx}\,dx.$$

$\Phi(s)$ is holomorphic for $\sigma < 0$, $\Psi(s)$ holomorphic for $\sigma > 0$.

We now replace y by $y + \xi$ in (10), and then multiply both members by $e^{-\sigma|y| + i\tau(y+\xi)}\,dy$, $\sigma > 0$, and finally integrate over $-\infty < y < \infty$. In this way we get

$$(11) \qquad \int_0^\infty k(x)\, e^{-i\tau x} U_\varphi(\sigma, \tau, x+\xi)\, dx = U_\psi(\sigma, \tau, \xi),$$

where

$$U_\varphi(\sigma, \tau, \xi) = \int_0^\infty \varphi(y)\, e^{-\sigma|y-\xi| + i\tau y}\, dy,$$

$$U_\psi(\sigma, \tau, \xi) = \int_{-\infty}^0 \psi(y)\, e^{-\sigma|y-\xi| + i\tau y}\, dy.$$

For $\xi \leq 0$, we have $U_\varphi(\sigma, \tau, \xi) = e^{\sigma\xi}\, \Phi(-\sigma + i\tau)$, and for $\xi \geq 0$, $U_\psi(\sigma, \tau, \xi) = e^{-\sigma\xi}\, \Psi(\sigma + i\tau)$. From this and (11) it now follows that the function

$$V(\sigma, \tau, \xi) = K(\sigma + i\tau)\, U_\varphi(\sigma, \tau, \xi) - U_\psi(\sigma, \tau, \xi)$$

satisfies the equation

$$(12) \qquad \int_0^\infty k(x)\, e^{-i\tau x} V(\sigma, \tau, x+\xi)\, dx = 0, \text{ for } \xi \geq 0.$$

Let $M(\sigma, \tau) = \underset{\xi \geq 0}{\text{l.u.b.}} |V(\sigma, \tau, \xi)|$. In estimating this quantity we use a method due to Beurling [5]. We first observe that there exists a constant m, such that $\left|\frac{\partial V}{\partial \xi}\right| \leq m$ for all σ, τ and ξ. By use of (12) we get

$$(13) \qquad |K(i\tau)|\, M(\sigma, \tau) \leq m \int_0^a x\, |k(x)|\, dx + 2\, M(\sigma, \tau) \int_a^\infty |k(x)|\, dx.$$

Suppose now that $i\tau$ does not $\epsilon\, \Lambda_\varphi$. It is then possible to choose the function $k(x)$ such that $K(i\tau) \neq 0$. By choosing a big enough, we can make the last integral in (13) $< \frac{1}{4} |K(i\tau)|$, and it results that $M(\sigma, \tau)$ is bounded for $\sigma > 0$.

In particular,

$$V(\sigma, \tau, 0) = K(\sigma + i\tau)\, \Phi(-\sigma + i\tau) - \Psi(\sigma + i\tau)$$

is bounded for $\sigma > 0$. As $K(s)$ is continuous in $\sigma \geq 0$, $K(i\tau) \neq 0$ implies that $|K(it)| > \delta > 0$ in an interval $|t - \tau| < \eta$, from which it follows that $V(\sigma, t, 0)$ is uniformly bounded for $\sigma > 0$ and t belonging to this interval. The function

$$\Phi(-\sigma + it) - \frac{\Psi(\sigma + it)}{K(\sigma + it)} \tag{14}$$

which is harmonic in a rectangle $0 < \sigma < \sigma_0$, $|t - \tau| < \eta$, is therefore bounded in a rectangle $0 < \sigma < \sigma_1 < \sigma_0$, $|t - \tau| < \eta$, which implies the existence of boundary values a.e. on the interval $|t - \tau| < \eta$ on the imaginary axis. Thus

$$\lim_{\sigma \to +0} V(\sigma, t, 0) = V(t)$$

exists and is bounded a.e. on the interval in question.

Let now $H(t)$ be a function, defined in the following way: $H(\tau) = 1$, $H(t) = 0$ for $|t - \tau| \geq \eta$, and is linear in $(\tau - \eta, \tau)$ and $(\tau, \tau + \eta)$. Obviously

$$H(t) = \int_{-\infty}^{\infty} h(x)\, e^{-itx}\, dx,$$

where $h(x) \in L^1(-\infty, \infty)$. Setting

$$v(y, \sigma) = e^{-\sigma y} \int_0^{\infty} k(x)\, \varphi(x + y)\, e^{-2\sigma x}\, dx - e^{\sigma y}\, \psi(y), \tag{15}$$

we get

$$\frac{1}{2\pi} \int_{\tau - \eta}^{\tau + \eta} H(t)\, V(\sigma, t, 0)\, e^{-itx}\, dt = \int_{-\infty}^{\infty} h(y - x)\, v(y, \sigma)\, dy. \tag{16}$$

We observe that $v(y, \sigma)$ is uniformly bounded for $\sigma \geq 0$ and all y, and that $\lim_{\sigma \to +0} v(y, \sigma) = 0$. Hence the right member of (16) has the limit 0 as $\sigma \to +0$. As $V(\sigma, t, 0)$ is uniformly bounded for $\sigma > 0$ on the interval $|t - \tau| < \eta$, we can pass to the limit under the integral sign in the left member of (16), obtaining

$$\frac{1}{2\pi} \int_{\tau-\eta}^{\tau+\eta} H(t)\, V(t)\, e^{-itx}\, dt = 0,$$

from which it follows that $V(t) = 0$ a.e. on the interval in question.

The function (14) thus has boundary values $= 0$ a.e. on the interval and is bounded in the vicinity of it. Consequently, we can apply Lemma 2, which shows that $\Phi(s)$ can be analytically continued over the interval into the right half-plane by the function $\Psi(s)/K(s)$.

Let us now assume that Λ_φ is empty. This implies that $\Phi(s)$ can be analytically continued over the entire imaginary axis, the continuation being $\Psi(s)/K(s)$, where $K(s)$ is the Laplace transform of an arbitrary function $k(x) \not\equiv 0$ in $L^1(0, \infty)$ which satisfies (10) and $\Psi(s)$ depends on $k(x)$ as above. $\Psi(s)/K(s)$ must be regular for $\sigma > 0$, for, if s_0 were a pole, $K(s_0) = 0$ must hold for all functions $k(x)$ of the above kind, which would imply that $s_0 \in \Lambda_\varphi$, contrary to our assumption. Consequently, if Λ_φ is empty, $\Phi(s)$ must be an entire function.

Let us first assume that $k(x)$ is not $= 0$ a.e. in any interval $(0, a)$, $a > 0$. Using Lemmas 3 and 4, we get

$$\iint_{|s| < R,\, \sigma > 0}^{+} \log \left| \frac{1}{K(s)} \right| d\sigma\, dt = o(R^3).$$

Since $|\Phi(s)| = O\left(\frac{1}{|\sigma|}\right)$ for $\sigma < 0$ and $|\Psi(s)| = O\left(\frac{1}{\sigma}\right)$ for $\sigma > 0$, it results that

$$\iint_{|s| < R}^{+} \log |\Phi(s)|\, d\sigma\, dt = o(R^3).$$

Setting $M(r) = \max_{|s|=r} |\Phi(s)|$, we obtain from this, by aid of the Poisson integral,

$$\int_0^R r \log M(r)\, dr = o(R^3),$$

or, $M(r)$ being nondecreasing,

$$\log M(r) = o(r).$$

Since $\Phi(s)$ is bounded on every line $\sigma = \sigma_0 < 0$, the Phragmén-Lindelöf theorem yields that $\Phi(s)$ is bounded in the entire plane, and thus is a constant, which must obviously be $= 0$. Hence $\varphi(x) = 0$ a.e.

If $A > 0$ is the largest number such that $k(x) = 0$ a.e. in $(0, A)$, we set $k_A(x) = k(x + A)$ and $\varphi_A(x) = \varphi(x + A)$. (10) can now be written

$$\int_0^\infty k_A(x)\, \varphi_A(x + y)\, dx = \psi(y).$$

$\varphi(x)$ having an empty spectral set, the same must be true for $\varphi_A(x)$. Since $k_A(x)$ is not $= 0$ a.e. in any interval $(0, a)$, $a > 0$, it follows in the same way as above that $\varphi_A(x) = 0$ a.e., or $\varphi(x) = 0$ a.e. for $x > A$. This completes the proof of Theorem 1.

Note: A function $\varphi(x)$ in $L^\infty(0, \infty)$, of which a translation $\varphi(x + A)$, $A > 0$, is $= 0$ a.e., may be regarded as a "degenerated eigen-function" e^{-sx}, corresponding to the point ∞ of Λ_φ. Having introduced this concept of degenerated eigen-functions, we see that Theorem 1 asserts that Λ_φ is never empty, provided $\varphi(x)$ not $= 0$ a.e.

10. Some properties of the transform $\Phi(s)$ of a mean-periodic function in $L^\infty(0, \infty)$. — From the above proof certain conclusions can be drawn concerning the function $\Phi(s)$ in the case when $\varphi(x)$ is mean-periodic. We formulate these conclusions in the following theorem:

Theorem 2. *Let* $\varphi(x) \in L^\infty(0, \infty)$ *and be mean-periodic, and let* $\Phi(s)$ *be its Laplace transform, defined as above. Then* $\Phi(s)$ *is a single-valued analytic function, regular in the entire plane with the possible exception of singularities on the imaginary axis, forming at most a set of measure zero, and poles* $\{s_n\}_1^\infty$ *in the half-plane* $\sigma > 0$, *with*

$$\sum_{n=1}^{\infty} \frac{\mathrm{Re}\,(s_n)}{1+|s_n|^2} < \infty .$$

The set of singularities of $\Phi(s)$ *is identical with* Λ_φ; *if* $s_0 = \sigma_0 + i t_0 \in \Lambda_\varphi$, $\sigma_0 > 0$, *of the multiplicity* q, *then* s_0 *is a pole of* $\Phi(s)$ *of the same multiplicity.*

Proof: $\varphi(x)$ being mean-periodic, $L^1(0, \infty)$ contains a function $k(x) \not\equiv 0$ satisfying (10). As has just been proved, $\Phi(s)$ can be analytically continued over an interval on the imaginary axis around every point $i\tau$ such that $K(i\tau) \neq 0$. As $K(it) = 0$ for at most a set of measure zero, it is easily seen that $\Phi(s)$ is single-valued. The properties stated above about the set of singularities follows immediately from this and the fact that $\Phi(s)$ in the right half-plane can be written in the form $\Psi(s)/K(s)$.

It remains to show that the set of singularities is identical with Λ_φ.

To this end, let us first suppose that s_0 is a singularity of $\Phi(s)$. If $\sigma_0 = 0$, s_0 must obviously belong to Λ_φ. Suppose now that $\sigma_0 > 0$ and that s_0 is a pole of multiplicity m. Let $k(x)$ be an arbitrary function in $L^1(0, \infty)$ satisfying (10). By the representation of $\Phi(s)$ for $\sigma > 0$, $K(s)$ must have a zero at s_0 of multiplicity m at least, and it follows that $q \geq m$.

Conversely, suppose that $s_0 \in \Lambda_\varphi$, and let us first assume that $\sigma_0 > 0$. In the representation $\Psi(s)/K(s)$ of $\Phi(s)$, $K(s)$ must have a zero at s_0 of multiplicity $\lambda \geq q$. Suppose that s_0 is a zero of $\Psi(s)$ of multiplicity μ, and let $\nu = \min(\lambda, \mu)$. The function

$$k_\nu(x) = \frac{e^{s_0 x}}{(\nu-1)!} \int_x^\infty (y-x)^{\nu-1} k(y)\, e^{-s_0 y}\, dy, \quad \nu \geq 1,$$

$k_0(x) = k(x)$, belongs to $L^1(0, \infty)$ and has the Laplace transform $K(s)/(s_0 - s)^\nu$ and satisfies (10). Thus $\lambda - \nu \geq q$, which yields $\nu = \mu$ and $\lambda - \mu = m \geq q$, i.e. $m = q$.

Suppose now that s_0 is a point of Λ_φ with $\sigma_0 = 0$. We must show that $\Phi(s)$ is singular at s_0. To this end, we shall avail ourselves of the following theorem, given by BEURLING in a lecture at Harvard university in 1949:

Let D be a linear subset, invariant under translation, of the space of bounded continuous functions on $(-\infty, \infty)$. Let $\overline{D}$ be the weak closure of D (defined as on page 8), and D^* be the closure of D in a new topology, defined as follows: $f(x) \in D^*$ if for every positive continuous function $\omega(x)$ which $\to \infty$ as $x \to \pm\infty$, there exists a function $g(x) \in D$, such that $|f(x) - g(x)| < \omega(x)$ for $-\infty < x < \infty$. Then $\overline{D} \equiv D^*$.

Here we are only concerned with functions defined on $(0, \infty)$ and with translations to the left; however, it is not hard to show that the above theorem still holds, mutatis mutandis.

Let us assume that $s_0 = 0$, which is no essential restriction. Suppose now that $\Phi(s)$ is holomorphic at the origin. Under these assumptions, we can construct from $\varphi(x)$ a function $\varphi_1(x)$ which is bounded and continuous on $(0, \infty)$, mean-periodic and has a transform $\Phi_1(s)$ holomorphic at the origin, and finally is such that $0 \in \Lambda_{\varphi_1}$. We may, for instance, take $\varphi_1(x) = \int_0^\infty \varphi(y + x)\, e^{-y}\, dy$.

Let $k(x) \not\equiv 0$ be a function in $L^1(0, \infty)$, orthogonal to $\varphi_1(x)$ in the sense of (10). Choose now a function $\omega(x)$, positive and continuous on $(0, \infty)$, which increases monotonically to ∞ as $x \to \infty$, and satisfies the conditions $\omega(x) < 1 + x$ and

$$\int_0^\infty |k(x)|\, \omega(x)\, dx < \infty. \tag{17}$$

As D we take the linear manifold consisting of all finite linear combinations of translations of $\varphi_1(x)$. From the above theorem follows the existence of a sequence $\{g_n(x)\}_1^\infty$ of functions in D such that

$$|g_n(x)-1|<\frac{\omega(x)}{n^2},\ x>0, \tag{18}$$

the constant 1 being included in the weak closure of D. Let $G_n(s)$ be the transform of $g_n(x)$. We have, $\sigma<0$,

$$\left|G_n(s)+\frac{1}{s}\right|=\left|\int_0^\infty (g_n(x)-1)\,e^{sx}\,dx\right|<\frac{1}{n^2}\int_0^\infty \omega(x)\,e^{\sigma x}\,dx<\frac{1}{n^2}\left(\frac{1}{\sigma^2}-\frac{1}{\sigma}\right),$$

which yields

$$\left|G_n\left(-\frac{1}{n}\right)\right|>n-2. \tag{19}$$

Let now Γ be a circle with its centre at the origin, and $\gamma=\gamma^+ +\gamma^-$ its periphery (γ^+ lying in $\sigma>0$, γ^- in $\sigma<0$). It is obviously possible to choose Γ in such a fashion that $\Phi_1(s)$ and hence all $G_n(s)$ are holomorphic in the closed circle, and $|K(s)|>\delta>0$ on γ^+. From (18) it follows that all $|G_n(s)|<\frac{M}{\sigma^2}$ on γ^-. Since $g_n(x)$ is orthogonal to $k(x)$ in the sense of (10), $G_n(s)$ can be written in the form $\Psi_n(s)/K(s)$ in $\sigma>0$. Using (17), it is easy to see that all $|\Psi_n(s)|<\frac{M}{\sigma}$ on γ^+. Thus $|G_n(s)|<\frac{M}{\sigma^2}$ on γ, where M is independent of n, from which it follows in an obvious way that $|G_n(s)|$ is uniformly bounded for all n in a circle with, say, half the radius of that of Γ. This is a contradiction to (19), which shows that $\Phi_1(s)$ and thus $\Phi(s)$ is singular at the origin.

11. On the singularities of $\Phi(s)$ on the imaginary axis. — Let $\varphi(x)$ be a mean-periodic function in $L^\infty(0,\infty)$. In connection with Theorem 2, the following question presents itself: If $\Phi(s)$ has singularities on the imaginary axis, what can be said about them? It may seem natural to surmise that an isolated singularity on the imaginary axis must be a pole (and then, of course, of multiplicity 1). However, this is not always the case, as will now be shown by an example.

We consider the simple Bessel function

$$\varphi(x) = J_0(2\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n!)^2}, \quad x \geq 0.$$

$\varphi(x) \in L^\infty(0, \infty)$ and has the transform

$$\Phi(s) = -\frac{1}{s} e^{\frac{1}{s}},$$

which is holomorphic in the entire plane with the exception of an essential singularity at the origin. We shall now show that $\varphi(x)$ is mean-periodic, its spectral set thus consisting of the origin alone.

To this end, we write, for $\sigma > 0$, $\Phi(s) = \Psi(s)/K(s)$, where

$$\Psi(s) = -\frac{s}{(1+s)^3} \text{ and } K(s) = \frac{s^2}{(1+s)^3} e^{-\frac{1}{s}}.$$

$\Psi(s)$ is the transform (as defined above) of the function $\psi(x) = 0$ for $x \geq 0$ and $= \left(\frac{x^2}{2} + x\right) e^x$ for $x < 0$. By the inequalities $|K(s)| < 1/\sqrt{1+t^2}$ and $|K'(s)| < 4/(1+t^2)$, $\sigma > 0$, $K(s)$ is the Laplace transform of a function $k(x)$, such that $k(x)$ and $x\,k(x) \in L^2(0, \infty)$, i.e. $k(x) \in L^1(0, \infty)$.

Applying Parseval's relation, we get, $\sigma > 0$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |K(\sigma + i\,t)\,\Phi(-\sigma + i\,t) - \Psi(\sigma + i\,t)|^2\,dt = \int_{-\infty}^{\infty} |v(y, \sigma)|^2\,dy,$$

where $v(y, \sigma)$ is defined by (15). As the integrand in the left member is $< 2/(1+t^2)$ for $\sigma > 0$, it follows, by letting $\sigma \to +0$, that

$$v(y, 0) = \int_0^{\infty} k(x)\,\varphi(x+y)\,dx - \psi(y) \equiv 0,$$

i.e. $\varphi(x)$ is mean-periodic.

12. On spectral synthesis. — Spectral synthesis of a function $\varphi(x)$ in $L^\infty(0,\infty)$ is in general not possible, i.e. $\varphi(x)$ is not always contained in the weak closure C'_φ of the linear manifold spanned by $\{x^\nu e^{-sx}\}$, $s \in \Lambda_\varphi$, $\nu = 0, \ldots, q-1$ (q = mult. of s). This results immediately from the fact that the addition to $\varphi(x)$ of an arbitrary "degenerated eigen-function" does not change its spectral set. (Now, we do not count the point at infinity as a possible point of the spectral set.) In view of this, it seems natural to ask if the spectral synthesis might be possible in the sense that to every mean-periodic $\varphi(x)$ corresponds a constant $A \geq 0$, such that $\varphi(x+A) \in C'_\varphi$. That this is not so in general is immediately seen from the above example $\varphi(x) = J_0(2\sqrt{x})$. However, in the simple case when Λ_φ is entirely situated in the open half-plane $\sigma > 0$, the spectral synthesis is always possible in this sense, as we are now going to prove:

Let us suppose that Λ_φ has no point on the imaginary axis. We have to show that there exists a constant $A \geq 0$ such that

$$\int_0^\infty h(y)\,\varphi(y+A)\,dy = 0,$$

for every $h(x) \in L^1(0,\infty)$ having a Laplace transform $H(s)$, which is zero (of the proper multiplicity) for every $s \in \Lambda_\varphi$. Let us set

$$f(x) = \int_0^\infty h(y)\,\varphi(y+x)\,dy, \quad x \geq 0,$$

$f(x) \equiv 0$, $x < 0$. If $k(x)$ is a function in $L^1(0,\infty)$ orthogonal to $\varphi(x)$ in the sense of (10), then $k(x)$ is also orthogonal to $f(x)$. By an easy computation, it follows from this fact that the transform $F(s)$ of $f(x)$ can be represented in $\sigma > 0$ by the function

$$H(s)\,\frac{\Psi(s)}{K(s)} - \int_0^\infty h(y)\,e^{-sy}\,dy \int_0^y \varphi(x)\,e^{sx}\,dx,$$

which is regular for $\sigma > 0$. Since Λ_φ has no point on the imaginary axis, the same must be true for Λ_f. Hence $F(s)$ is an entire function. Choosing $k(x)$ such that the interval $(0, A)$ in which it is zero a.e. is as small as possible, we see that $f(x) = 0$ for $x \geq A$, and our assertion is proved.

13. Closure theorems in $L^1(0, \infty)$. — The following closure theorem is an immediate consequence of Theorem 1:

Theorem 3. *Let $k(x) \in L^1(0, \infty)$, $k(x) \equiv 0$ for $x < 0$. The set $k(x - \tau)$, $\tau \geq 0$, is fundamental on $L^1(0, \infty)$ if and only if $k(x)$ is not $= 0$ a.e. in any interval $(0, A)$, $A > 0$, and its Laplace transform $K(s)$ is $\neq 0$ in the closed half-plane $\sigma \geq 0$.*

Proof: The conditions are obviously necessary. Let $\varphi(x) \in L^\infty(0, \infty)$ and be such that

$$\int_0^\infty k(x - y)\, \varphi(x)\, dx = \psi(y),$$

where $\psi(y) \equiv 0$ for $y \geq 0$. This relation may be written in the form (10), and Theorem 1 yields that $\varphi(x) = 0$ a.e.

Using the concept of inner factor, it will be seen that the conditions of Theorem 3 imply that the inner factor of $K(s)$ is $\equiv 1$. However, this last condition is not sufficient, as it does not necessarily imply that $K(s) \neq 0$ on the imaginary axis.

In precisely the same way as above, the following more general assertion is established:

Theorem 4. *Let $\{k(x)\}$ be a family of functions in $L^1(0, \infty)$, each function being $\equiv 0$ for $x < 0$. The set $\{k(x - \tau)\}$, $\tau \geq 0$, is fundamental on $L^1(0, \infty)$ if and only if at least one function is not $= 0$ a.e. in any interval $(0, A)$, $A > 0$, and the Laplace transforms $\{K(s)\}$ have no common zero in $\sigma \geq 0$.*

14. Closure theorems in $L^2(0, \infty)$. — We now turn our attention to the Hilbert space $L^2(0, \infty)$. This case has been treated by Karhunen [11], who has proved the analogon of Theorem 3 in $L^2(0, \infty)$. This theorem can be stated in the following form, using the concept of inner factor:

Let $k(x) \in L^2(0, \infty)$, $k(x) \equiv 0$ *for* $x < 0$. *The set* $k(x - \tau)$, $\tau \geq 0$, *is fundamental on* $L^2(0, \infty)$ *if and only if the inner factor of the Laplace transform* $K(s)$ *of* $k(x)$ *is* $\equiv 1$.

It follows immediately from this theorem that the analogon of Theorem 1 is not valid for functions $\varphi(x)$ in $L^2(0, \infty)$. It ought to be pointed out that this difference between the cases $L^\infty(0, \infty)$ and $L^2(0, \infty)$ is essentially due to the fact that the numbers s occurring in the eigen-functions e^{-sx} of the translations to the left in the first case fill the closed half-plane $\sigma \geq 0$, but in the second case only the open half-plane $\sigma > 0$.

If we make use of a method due to Beurling [7], it is easy to state the above theorem in a somewhat more precise form:

Let us denote by C_k^* the closure of the linear manifold spanned by $k(x - \tau)$, $\tau \geq 0$, and denote by $K_0(s)$ and $K_1(s)$ the inner and outer factors of $K(s)$. Let $h(x)$ be another function in $L^2(0, \infty)$, and $H(s) = H_0 H_1$ its Laplace transform.

Theorem 5. *A necessary and sufficient condition for the relation* $h \in C_k^*$ *is that* K_0 *is a divisor of* H_0.

Proof: $K_1(s)$ is the Laplace transform of a function $k_1(x) \in L^2(0, \infty)$. If we set

$$g(x) = \underset{N \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-N}^{N} \frac{H(it)}{K_0(it)} e^{itx}\, dt,$$

Parseval's relation yields, $\tau_n \geq 0$,

$$\int_0^\infty \Big| \sum_n A_n k(x - \tau_n) - h(x) \Big|^2 dx = \tag{20}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big| K(it) \sum_n A_n e^{-it\tau_n} - H(it) \Big|^2 dt =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| K_1(it) \sum_n A_n e^{-it\tau_n} - \frac{H(it)}{K_0(it)} \right|^2 dt =$$

$$= \int_0^\infty \Big| \sum_n A_n k_1(x - \tau_n) - g(x) \Big|^2 dx + \int_{-\infty}^{0} |g(x)|^2 dx.$$

If $h \in C_k^*$, it follows from (20) that $g(x) = 0$ a.e. for $x < 0$. Hence, $G(s)$ denoting the Laplace transform of $g(x)$, it is seen that the function $K_0(s)\,G(s) - H(s)$ has the boundary values 0 a.e. on the imaginary axis, and therefore is $\equiv 0$. This implies that K_0 is a divisor of H_0. — On the other hand, if K_0 is a divisor of H_0, then $g(x) = 0$ a.e. for $x < 0$. Since $K_1(s)$ has the inner factor 1, it follows from KARHUNEN's theorem that (20) can be made arbitrarily small by choosing the A_n and τ_n suitably, and the theorem follows.

Theorem 6. *Let $k(x)$ and $h(x)$ be two functions in $L^2(0, \infty)$, both of them $\equiv 0$ for $x < 0$, and let $C_{k,h}^*$ be the closure of the linear manifold spanned by the set $k(x - \tau)$, $h(x - \tau)$, $\tau \geq 0$. Let F_0 be the largest common divisor of K_0 and H_0, and set*

$$f(x) = \underset{N \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-N}^{N} \frac{F_0(it)}{1 + it}\, e^{itx}\, dt.$$

Then $C_{k,h}^* \equiv C_f^*$.

Proof: Let $g(x) \in L^2(0, \infty)$ and $G(s) = G_0\, G_1$ be its Laplace transform (the subscripts 0 and 1 will here always indicate an inner and an outer factor, respectively). Let us set $K = K_0 K_1 = F_0 P_0 K_1 = F_0 P$ and $H = H_0 H_1 = F_0 Q_0 H_1 = F_0 Q$, and let P and Q be the transforms of $p(x)$ and $q(x)$, which both belong to $L^2(0, \infty)$. If τ_n and $\tau_n' \geq 0$, and

$$r(x) = \underset{N \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-N}^{N} \frac{G(it)}{F_0(it)}\, e^{itx}\, dt,$$

we get, in complete analogy to (20):

$$(21) \qquad \int_0^\infty \Big| \sum_n A_n\, k(x - \tau_n) + \sum_n B_n\, h(x - \tau_n') - g(x) \Big|^2\, dx =$$

$$= \int_0^\infty \Big| \sum_n A_n\, p(x - \tau_n) + \sum_n B_n\, q(x - \tau_n') - r(x) \Big|^2\, dx + \int_{-\infty}^0 |r(x)|^2\, dx.$$

Suppose now that $g \in C^*_{k,h}$. Then $r(x) = 0$ a.e. for $x < 0$, from which it follows in the same way as above that F_0 is a divisor of G_0, or $g \in C^*_f$, i.e. $C^*_{k,h} \subset C^*_f$.

P_0 and Q_0 have the largest common divisor 1. It remains to prove that this implies that $C^*_{p,q} \equiv L^2(0, \infty)$. For it will then follow from (21) (with $g = f$) that $f \in C^*_{k,h}$, i.e. $C^*_f \subset C^*_{k,h}$.

Let us suppose that $C^*_{p,q}$ is a proper subset of $L^2(0, \infty)$. Then there exists a $\varphi(x) \not\equiv 0$ in $L^2(0, \infty)$, such that

$$(22) \qquad \int_0^\infty p(x-y)\,\varphi(x)\,dx = \gamma(y), \quad \int_0^\infty q(x-y)\,\varphi(x)\,dx = \delta(y),$$

where $\gamma(y)$ and $\delta(y) = 0$ for $y \geq 0$. Let Φ, Γ and Δ be the Laplace transforms of φ, γ and δ (with s replaced by $-s$). From (22) it follows by Parseval's relation that $P(it)\,\Phi(it) = \Gamma(it)$ and $Q(it)\,\Phi(it) = \Delta(it)$ a.e. The function $P(s)\,\Delta(s) - Q(s)\,\Gamma(s)$ which is regular for $\sigma > 0$ is thus $\equiv 0$. Hence $P_0\,\Delta_0 \equiv Q_0\,\Gamma_0$, from which it follows that P_0 is a divisor of Γ_0.

Consequently, the function $\Gamma(s)/P(s)$ is regular for $\sigma > 0$, and by an argument similar to that on page 17, we can conclude that

$$\log\left|\frac{\Gamma(s)}{P(s)}\right| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma\,\log|\Phi(iy)|}{\sigma^2 + (y-t)^2}\,dy,$$

and that the Fourier transform $\varphi(-x)$ of $\Phi(it)$ is $= 0$ a.e. for $x < 0$. This contradiction proves the theorem.

15. An application to the Riemann ζ-function. — As an application of Theorem 6, we shall now prove the following assertion:

Theorem 7. *Let*

$$(23) \qquad K(x, a) = \left[\frac{a}{x}\right] - a\left[\frac{1}{x}\right], \quad 0 < a \leq 1,$$

(where [ ] denotes the integral part). A necessary and sufficient condition for the Riemann ζ-function to be $\neq 0$ for $\sigma > \frac{1}{2}$ is that the set (23) *is fundamental on* $L^2(0, 1)$.

Proof: Let $f(x)$ be a function in $L^2(0,1)$, satisfying

$$\int_0^1 K(x,\alpha) f(x)\, dx = 0,\ 0<\alpha\leq 1.$$

This relation is equivalent to

$$\int_0^1 K(x,\alpha) f(\beta x)\, dx = 0,\ 0<\alpha\leq 1,\ 0<\beta\leq 1, \tag{24}$$

for the substitution $y=\beta x$ transforms the last integral into

$$\frac{1}{\beta}\int_0^1 K(y,\alpha\beta) f(y)\, dy - \frac{\alpha}{\beta}\int_0^1 K(y,\beta) f(y)\, dy.$$

Setting $x=e^{-y}$, $\alpha=e^{-\xi}$ and $\beta=e^{-\eta}$ in (24), we obtain

$$\int_0^\infty h_\xi(y-\eta)\,\varphi(y)\, dy = 0,\ \xi\geq 0,\ \eta\geq 0,$$

where

$$h_\xi(y) = ([e^{y-\xi}] - e^{-\xi}[e^y])\, e^{-\frac{y}{2}},\ \varphi(y) = f(e^{-y})\, e^{-\frac{y}{2}},$$

these functions both belonging to $L^2(0,\infty)$. $h_\xi(y)=0$ for $y<0$, $\xi\geq 0$. The Laplace transform of $h_\xi(y)$ is

$$H_\xi(s) = (e^{-(s+\frac{1}{2})\xi} - e^{-\xi})\,\frac{\zeta\left(s+\frac{1}{2}\right)}{s+\frac{1}{2}}.$$

$H_\xi(s)$ has the pole $s=-\frac{1}{2}$ as the only singularity, and, for $\xi>0$, $h_\xi(y)$ is not $=0$ in any interval $(0,A)$. From this it follows that the inner factor of H_ξ is a Blaschke product.

Besides possible zeros of $\zeta\left(s+\frac{1}{2}\right)$, H_ξ has, for $\sigma>0$, the zeros $\frac{1}{2}+\frac{2n\pi}{\xi}i$, $n=\pm 1, \pm 2, \ldots$ Now, let us also take a function $h_{\xi'}(y)$, where ξ and ξ' are incommensurable. Then H_ξ and $H_{\xi'}$ have only the zeros of $\zeta\left(s+\frac{1}{2}\right)$ in common, and our assertion follows from Theorem 6.

CHAPTER 3

ON TRANSLATIONS OF FUNCTIONS IN L_p^1 AND L_p^∞

16. Definitions. — In this chapter we shall study, in some special cases, translations of functions in a space L_p^1 and L_p^∞. The subscript p now denotes a weight-function $p(x)$ of the kind introduced by BEURLING in [1], and L_p^1 is the space of all measurable functions $k(x)$ defined on $(-\infty, \infty)$, with the norm

$$\int_{-\infty}^{\infty} |k(x)|\, p(x)\, dx < \infty,$$

while L_p^∞ consists of all measurable functions $\varphi(x)$, with the norm

$$\text{l.u.b.}_{-\infty<x<\infty} \frac{|\varphi(x)|}{p(x)} < \infty.$$

A weight-function of the above-mentioned kind is characterized by the conditions $p(x) \geq p(0) = 1$, $p(x+y) \leq p(x)\, p(y)$, and $p(\varrho x) \geq p(x)$, for $\varrho > 1$. These relations imply the existence of the two limits

$$\alpha = \lim_{x\to\infty} \frac{\log p(x)}{x}, \quad \beta = \lim_{x\to-\infty} \frac{\log p(x)}{-x}.$$

One of the numbers α and β can be allowed to be $= \infty$, but not both of them.

Beurling has classified the weight-functions in the following way: If $\alpha+\beta>0$, $p(x)$ is said to be of analytic type. If $\alpha+\beta=0$, $p(x)$ is of non-analytic type, and in this case $p(x)$ is said to be of quasianalytic type if the integral

$$\int_{-\infty}^{\infty} \frac{\log\, p(x)}{1+x^2}\, dx$$

diverges, and of non-quasianalytic type if this integral converges.

The spectral set Λ_φ of a function $\varphi(x)$ in L_p^∞ has already been defined (page 8), viz. as the set of all complex numbers s, such that the integral equation

$$\int_{-\infty}^{\infty} k(x)\,\varphi(x-y)\,dx = 0,\ -\infty<y<\infty, \tag{25}$$

where $k(x)\in L_p^1$, implies that

$$\int_{-\infty}^{\infty} k(x)\, e^{-sx}\, dx = 0.$$

In the analytic case, Λ_φ is a closed set situated in the closed strip $S_{\alpha,\beta}$ defined by $-\alpha\leq\sigma\leq\beta$, and in the non-analytic case, Λ_φ is a closed set on the imaginary axis.

As has been mentioned in the introduction, Beurling has proved that if $\varphi(x)\in L_p^\infty$, where $p(x)$ is of non-quasianalytic type, then Λ_φ is not empty, provided that $\varphi(x)$ is not $=0$ a.e. Can this theorem be extended to spaces L_p^∞, where $p(x)$ is of analytic or quasianalytic type? The space $L^\infty(0,\infty)$ which has been studied in the preceding chapter, may, as we have seen (page 9), be regarded as a space L_p^∞, where $p(x)$ is of analytic type. We have seen that in this space there exist functions, not $=0$ a.e., having empty spectral sets (if we do not count the point at infinity as a possible point of the spectral set). We shall now show, by means of examples, that this

situation prevails in other spaces L_p^∞, where $p(x)$ is of analytic or quasianalytic type (and where we are concerned with translations in both directions). Our results are very incomplete, only particular weight-functions being considered.

17. The closure theorem in a space L_p^1, where $p(x)$ is of analytic type. — Let us consider the weight-function

$$p(x) = \begin{cases} e^{ax}, & x \geq 0, \\ e^{-\beta x}, & x < 0, \end{cases}$$

where a and $\beta \geq 0$, $0 < a + \beta < \infty$. In the space L_p^1 determined by this function, we shall prove the following closure theorem:

Theorem 8. *Let $k(x) \in L_p^1$ (with the above defined $p(x)$). The set $k(x-\tau)$, $-\infty < \tau < \infty$, is fundamental on L_p^1 if and only if the Laplace transform*

$$K(s) = \int_{-\infty}^{\infty} k(x)\, e^{-sx}\, dx$$

is $\neq 0$ in the closed strip $S_{a,\beta}$, and furthermore

$$\lim_{t \to \pm\infty} e^{-\frac{\pi|t|}{a+\beta}} \log |K(\sigma + it)| = 0, \quad -a < \sigma < \beta. \tag{26}$$

Proof: It is immediately seen that the condition $K(s) \neq 0$ in $S_{a,\beta}$ is necessary. It may be observed that the additional condition (26) means that $K(s)$ is not "too small" as $t \to \pm\infty$. We shall prove our assertion by the method used in the proof of Theorem 1, the argument here being somewhat more complicated.

Let us first prove the sufficiency of the stated conditions, i.e. that they imply that a function $\varphi(x) \in L_p^\infty$, satisfying (25), must be $= 0$ a.e.

$K(s)$ is holomorphic in the open strip $S_{a,\beta}$, and continuous in the closed strip. Let us set

$$\Phi^+(s) = -\int_{-\infty}^{0} \varphi(x)\, e^{sx}\, dx, \tag{27}$$

$$\Phi^-(s) = \int_{0}^{\infty} \varphi(x)\, e^{sx}\, dx. \tag{28}$$

$\Phi^+(s)$ is holomorphic in the half-plane $\sigma > \beta$, and $\Phi^-(s)$ is holomorphic in $\sigma < -\alpha$. Again, let

$$\psi(y) = \int_{y}^{\infty} k(x)\, \varphi(x-y)\, dx = -\int_{-\infty}^{y} k(x)\, \varphi(x-y)\, dx. \tag{29}$$

From these two representations we find that $|\psi(y)|\, p(y)$ is bounded, and $\to 0$ as $y \to \pm\infty$. Hence the function

$$\Psi(s) = \int_{-\infty}^{\infty} \psi(x)\, e^{-sx}\, dx \tag{30}$$

is holomorphic in the open strip $S_{\alpha,\beta}$.

Now we replace y by $y+\xi$ in the first of the relations (29), and multiply both members by $e^{\alpha y - (\sigma+\alpha)|y| - it(y+\xi)}\, dy$, where $-\alpha < \sigma < \beta$, and finally integrate over $(-\infty, \infty)$. In this manner we obtain

$$\int_{-\infty}^{\infty} k(x)\, e^{(\alpha - it)x}\, U_\varphi(\sigma, t, x-\xi)\, dx = U_\psi(\sigma, t, \xi), \tag{31}$$

where

$$U_\varphi(\sigma, t, \xi) = \int_{0}^{\infty} \varphi(y)\, e^{-(\sigma+\alpha)|y-\xi| - (\alpha - it)y}\, dy,$$

$$U_\psi(\sigma, t, \xi) = \int_{-\infty}^{\infty} \psi(y)\, e^{-(\sigma+\alpha)|y-\xi| + (\alpha - it)y}\, dy.$$

For $\xi \leq 0$, we have

$$U_\varphi(\sigma, t, \xi) = e^{(\sigma+\alpha)\xi}\ \Phi^-(-\sigma - 2\alpha + it).$$

Furthermore, for all ξ we have

$$U_\psi(\sigma, t, \xi) = e^{(\sigma+\alpha)\xi}\ \Psi(\sigma + it) + F(\sigma, t, \xi), \tag{32}$$

where

$$F(\sigma, t, \xi) = e^{-(\sigma+\alpha)\xi} \int_{-\infty}^{\xi} \psi(y)\, e^{(\sigma+2\alpha-it)y}\, dy - \tag{33}$$

$$- e^{(\sigma+\alpha)\xi} \int_{-\infty}^{\xi} \psi(y)\, e^{-(\sigma+it)y}\, dy.$$

Now let us set

$$V(\sigma, t, \xi) = K(\sigma + it)\, U_\varphi(\sigma, t, \xi) - U_\psi(\sigma, t, -\xi). \tag{34}$$

After an easy computation, we obtain, using (31) and (32),

$$\int_{-\infty}^{\infty} k(x)\, e^{(\alpha-it)x}\, V(\sigma, t, x - \xi)\, dx = G(\sigma, t, \xi), \tag{35}$$

where

$$G(\sigma, t, \xi) = K(\sigma + it)\, F(\sigma, t, \xi) - \tag{36}$$

$$- \int_{-\infty}^{\infty} k(x)\, e^{(\alpha-it)x}\, F(\sigma, t, \xi - x)\, dx.$$

Let us now assume that $-\alpha < \sigma < \dfrac{\beta - \alpha}{2}$. We can then conclude from (33) that $|F(\sigma, t, \xi)| < c$ for $\xi \leq 0$, and that $|F(\sigma, t, \xi)| < c(1 + \xi)\, e^{(\sigma+\alpha)\xi}$ for $\xi > 0$. From these two inequalities and (36) it then follows that $|G(\sigma, t, \xi)| < c'$ for $\xi \leq 0$; c and c' are constants, independent of σ and t.

Let $M(\sigma, t) = \text{l.u.b.}_{\xi \geq 0} |V(\sigma, t, \xi)|$. Observing that $\left|\frac{\partial V}{\partial \xi}\right| < m$, where m is a constant which is independent of σ and t, and using (35), we obtain, for $a > 0$:

$$|K(-\alpha + it)| M(\sigma, t) \leq m \int_{-\infty}^{a} |x\, k(x)|\, e^{ax}\, dx +$$

$$+ 2\, M(\sigma, t) \int_{a}^{\infty} |k(x)|\, e^{ax}\, dx + c'.$$

$K(-\alpha + it)$ is $\neq 0$, by assumption. Consequently, we can now conclude, in precisely the same manner as on page 20, that $M(\sigma, t)$ is bounded as $\sigma \to -\alpha + 0$. In particular, setting $\xi = 0$ in (34), we see that

$$K(\sigma + it)\, \Phi^{-}(-\sigma - 2\alpha + it) - \Psi(\sigma + it) \tag{37}$$

is bounded for every fixed t as $\sigma \to -\alpha + 0$.

The Fourier transform of (37) is

$$v(y, \sigma) = e^{(\sigma + 2a)y} \int_{y}^{\infty} k(x)\, \varphi(x - y)\, e^{-2(\sigma + a)x}\, dx - e^{-\sigma y}\, \psi(y), \tag{38}$$

which is bounded, uniformly in y, and tends to zero as $\sigma \to -\alpha + 0$.

Using the above-mentioned properties of the functions (37) and (38), it now follows in the same way as on page 22 that $\Phi^{-}(s)$ can be analytically continued over the entire line $\sigma = -\alpha$ into the strip $S_{\alpha,\beta}$ by the function $\Psi(s)/K(s)$, which is regular in the strip. Similarly, $\Phi^{+}(s)$ can be analytically continued over the line $\sigma = \beta$, and its continuation in $S_{\alpha,\beta}$ is the same function $\Psi(s)/K(s)$.

Consequently, the functions $\Phi^{-}(s)$, $\Psi(s)/K(s)$ and $\Phi^{+}(s)$ are different representations of one and the same entire function, which we shall call $F(s)$.

The function $z = \exp\left(-\frac{\pi i}{2}\,\frac{2s+\alpha-\beta}{\alpha+\beta}\right)$ maps the strip $S_{\alpha,\beta}$ onto the half-plane $x \geq 0$, $z = x + iy = re^{i\Theta}$. The function $K(s)$ is thereby transformed into a function $K_1(z)$, which is regular, bounded and $\neq 0$ for $x > 0$. In view of this, we can write

$$\log|K_1(z)| = -Ax + \int_{-\infty}^{\infty} \frac{x(1+\xi^2)}{x^2+(\xi-y)^2}\,d\nu(\xi),$$

where $A \geq 0$ and $\int_{-\infty}^{\infty} |d\nu(\xi)| < \infty$. It is not difficult to see that the integral in the above formula is $o\left(\frac{r}{\cos\Theta}\right)$ as $r \to \infty$. Hence

$$\lim_{r\to\infty} \frac{1}{r} \log|K_1(re^{i\Theta})| = -A\cos\Theta, \quad |\Theta| < \frac{\pi}{2},$$

or

$$\lim_{t\to\infty} e^{-\frac{\pi t}{\alpha+\beta}} \log|K(\sigma+it)| = -A\cos\frac{\pi}{2}\,\frac{2\sigma+\alpha-\beta}{\alpha+\beta}, \tag{39}$$

$$-\alpha < \sigma < \beta.$$

The condition (26) (with $t \to +\infty$) is thus equivalent to $A = 0$.

By the above mapping, the part of $S_{\alpha,\beta}$ which lies in $t \geq 0$, maps into the domain $x \geq 0$, $r \geq 1$, in the z-plane, and the strip $-\frac{3\alpha+\beta}{2} < \sigma < \frac{\alpha+3\beta}{2}$, $t \geq 0$, goes into the domain $r \geq 1$, cut along the negative real axis. Let D be the part of this last domain which is situated to the right of the line $x = -1$. Furthermore, let $F_1(z)$ and $\Psi_1(z)$ be the functions into which $F(s)$ and $\Psi(s)$ are transformed.

In the s-plane, $|F(s)| < M/(-\sigma-\alpha)$ for $\sigma < -\alpha$, and $|F(s)| < M/(\sigma-\beta)$ for $\sigma > \beta$, and $|\Psi(s)| < M/(\sigma+\alpha)(\beta-\sigma)$ for $-\alpha < \sigma < \beta$. From these inequalities and the fact that $F_1(z) = \Psi_1(z)/K_1(z)$ for $x > 0$, together with the above-mentioned

properties of $K_1(z)$, it now readily follows that $|F_1(z)| < Mr$ on the boundary of D, and, by Lemmas 3 and 4, that the double integral of $\overset{+}{\log}|F_1(z)|$, extended over the common part of D and $r<R$, is $o(R^3)$ as $R\to\infty$. Hence

$$\lim_{r\to\infty}\frac{1}{r}\int_{-\Theta_r}^{\Theta_r}\overset{+}{\log}|F_1(re^{i\Theta})|\,d\Theta = 0,$$

$\Theta_r = \frac{\pi}{2} + \arcsin\frac{1}{r}$. Using the Phragmén-Lindelöf theorem, we can conclude from this that $|F_1(z)| < Mr$ in D, which yields that $|F(s)| < M\exp\left(\frac{\pi t}{\alpha+\beta}\right)$ in the half-plane $t\geq 0$.

Again, using (26) (with $t\to-\infty$), we get in the same manner as before that $|F(s)| < M\exp\left(-\frac{\pi t}{\alpha+\beta}\right)$ for $t\leq 0$. As $F(s)$ is bounded for $\sigma < -\alpha-\varepsilon$ and $\sigma > \beta+\varepsilon$, $\varepsilon>0$, $F(s)$ must be a constant, which obviously is zero. Hence $\varphi(x)=0$ a.e.

It remains to show that the condition (26) is necessary. Let us assume that $k(x)\in L_p^1$ and that $K(s)\neq 0$ in the closed strip $S_{\alpha,\beta}$, whereas the constant A in (39) is >0. It is not hard to see that such a function exists, and that

$$|K(s)| < M\left|\exp\left(-A\,e^{-\frac{\pi i}{2}\frac{2s+\alpha-\beta}{\alpha+\beta}}\right)\right| \tag{40}$$

in $S_{\alpha,\beta}$. We shall now construct a function $\varphi(x)\in L_p^\infty$ which is orthogonal to $k(x)$ in the sense of (25).

To this end, we set

$$G(s) = \frac{\exp\left(B\,e^{-\frac{\pi i}{2}\frac{2s+\alpha-\beta}{\alpha+\beta}}\right)}{(s+\alpha+1)\,(s-\beta-1)},\quad 0<B\leq A,$$

and

$$F(s) = \frac{1}{2\pi i}\int_\Gamma \frac{G(z)}{z-s}\,dz,\quad \sigma<-\alpha \text{ or } \sigma>\beta, \tag{41}$$

where Γ denotes the boundary of the strip $S_{\alpha,\beta}$ ($\sigma = -\alpha$ is described from $-\infty$ to $+\infty$, $\sigma = \beta$ in the opposite direction). $F(s)$ is an entire function, its representation in $S_{\alpha,\beta}$ being

$$F(s) = G(s) + \frac{1}{2\pi i}\int_{\Gamma} \frac{G(z)}{z-s}\,dz = G(s) + R(s). \tag{42}$$

Now let us set

$$\varphi(x) = \frac{1}{2\pi i}\int_{\Gamma} G(s)\,e^{-sx}\,ds. \tag{43}$$

Since $G(-\alpha+it)$ and $G(\beta+it)$ are summable, $\varphi(x) \in L_p^\infty$. Obviously, $\varphi(x)$ is not $=0$ a.e. By aid of $\varphi(x)$, the function $F(s)$ can be represented in the form (27) and (28) for $\sigma > \beta$ and $\sigma < -\alpha$, respectively.

In the strip $S_{\alpha,\beta}$ we write $F(s) = \Psi(s)/K(s)$, where $\Psi(s) = K(s)\,G(s) + K(s)\,R(s)$. Using (40) and the above representation of $R(s)$, it is easy to verify that $\Psi(s)$ can be written in the form (30), where $|\psi(x)|\,p(x)$ is bounded.

Let $-\alpha < \sigma < \beta$. The function

$$K(\sigma+it)\,\Phi^-(-\sigma-2\alpha+it) - \Psi(\sigma+it) = \\ = K(\sigma+it)\,(F(-\sigma-2\alpha+it) - F(\sigma+it))$$

has the Fourier transform (38), and is $< M/(1+t^2)$ as $\sigma \to -\alpha+0$. From this it follows, by the same argument as that used on page 27, that the functions $\varphi(x)$, $k(x)$ and $\psi(x)$ together satisfy the first of the relations (29). In the same way we can prove that they also satisfy the second of these two relations.

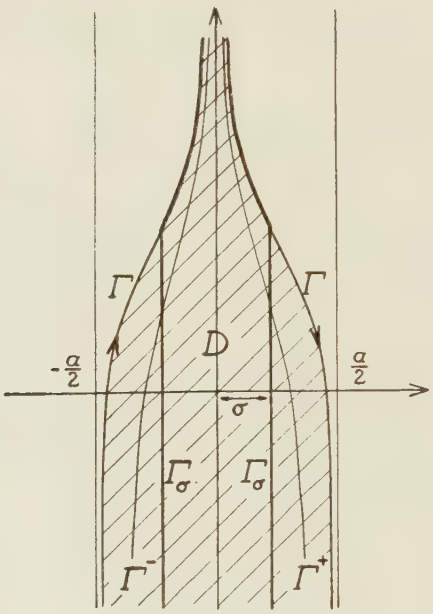
Consequently, the functions $\varphi(x)$ and $k(x)$ are orthogonal in the sense of (25), and Theorem 8 is established.

Obviously, the above constructed $\varphi(x)$ furnishes an example of a function in L_p^∞ having an empty spectral set.

18. Functions with empty spectral sets in spaces L_p^∞, where $p(x)$ is of quasianalytic type. — We shall now consider a space L_p^∞, where $p(x)$ is of quasianalytic type, and give a construction

of a function $\varphi(x) \in L_p^\infty$, which is not $= 0$ a.e., but has an empty spectral set.

To this end, let D be a domain of the type shown in the below figure:



The boundary Γ of D consists of two continuous curves $\sigma = \pm\, l(t)$ where $l(t)$ is a positive nonincreasing function with $l(-\infty) = \frac{a}{2} > 0$ and $l(\infty) = 0$.

Let $G_0(s)$ be a function which is holomorphic in D and continuous in $D + \Gamma$, and satisfies the following conditions: $|G_0(s)| > 1$ in D, $|G_0(s)| < M$ on Γ, $G(it)$ is real and $\to \infty$ as $t \to \infty$, while $G_0(s)$ tends to a constant c as $t \to -\infty$, uniformly in D.

Furthermore, we impose upon D and the associated function $G_0(s)$ the following condition: On the line $\mathrm{Re}(s) = \sigma$, $|G_0(s)|$ increases from c to a maximum $e^{\mu(\sigma)}$, attained for $t = \gamma(\sigma)$, and

then decreases; and the curves $t = \gamma(\sigma)$ (Γ^+ and Γ^- in the figure) are monotonically decreasing from $+\infty$ to $-\infty$ as $|\sigma|$ increases from 0 to $\frac{a}{2}$. ($\mu(\sigma)$ and $\gamma(\sigma)$ are obviously even functions.)

We may, for instance, take $l(t) = \frac{\pi}{2}$ for $t \leq 0$, $l(t) = \arcsin(e^{-t})$ for $t > 0$, and $G_0(s) = \exp\left\{\exp\left(\frac{\pi}{2} e^{-is}\right)\right\}$.

Now, we set $G(s) = G_0(s)/(a^2 - s^2)$ and define $F(s)$ for s outside D by formula (41), where Γ now denotes the boundary of D described in the direction indicated in the figure. It is obvious that $F(s)$ is an entire function $\not\equiv 0$, and that $F(s)$ is represented in D by formula (42).

Let us then define $\varphi(x)$ by (43). After having moved the path of integration to Γ_σ (see fig.), we obtain, $\omega = u + iv$,

$$|\varphi(x)| \leq \frac{1}{2\pi} \int_{\Gamma_\sigma} |G(\omega)|\, e^{|ux|}\, |d\omega| \leq M e^{\mu(\sigma) + \sigma|x|}.$$

From this it results that $|\varphi(x)| \leq M e^{q(x)}$, where

$$q(x) = \min_{0 < \sigma \leq \frac{a}{2}} (\mu(\sigma) + \sigma x) \tag{44}$$

for $x \geq 0$, and $q(-x) = q(x)$.

We can now write $F(s)$ in the form:

$$F(s) = \begin{cases} -\int_{-\infty}^{0} \varphi(x)\, e^{sx}\, dx, & \sigma > 0, \\ \int_{0}^{\infty} \varphi(x)\, e^{sx}\, dx, & \sigma < 0. \end{cases} \tag{45}$$

From (44) it immediately follows that, for $x > 0$, $q(x)$ is nondecreasing and concave, and that $q(x)/x$ decreases monotonically to zero as $x \to \infty$. By an elementary argument, we can

conclude from these properties that $q(x)$ is subadditive, i.e. that $q(x+y) \leq q(x) + q(y)$. Furthermore, the integral

$$\int_{-\infty}^{\infty} \frac{q(x)}{1+x^2}\, dx$$

must diverge. For, if this integral were convergent, the function $F(s)$, represented by (45) would be $\equiv 0$, according to a unicity theorem due to BEURLING ([3], page 13).

Hence $p(x) = e^{q(x)}$ is a weight-function of quasianalytic type, and $\varphi(x) \in L_p^\infty$. In the special example mentioned on page 45, we get a $p(x)$ such that $\log p(x) \sim \frac{\pi}{2} \frac{x}{\log x}$ as $x \to \infty$.

Let us now set $K(s) = \{G_0(s)\}^{-\alpha}/(a^2 - s^2)$, $\alpha > 1$, and

$$k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(it)\, e^{itx}\, dt.$$

If $x \geq 0$, we move the path of integration from the imaginary axis to the curve Γ^-, obtaining

$$|k(x)| \leq \frac{1}{2\pi} \int_{\Gamma^-} |K(s)|\, e^{\sigma x} |ds| \leq$$

$$\leq \exp\left\{ - \min_{-\frac{a}{2} \leq \sigma < 0} (\alpha \mu(\sigma) - \sigma x) \right\} \frac{1}{2\pi} \int_{\Gamma^-} \frac{|ds|}{|a^2 - s^2|} = M e^{-\alpha q\left(\frac{x}{\alpha}\right)}.$$

By moving the path of integration to Γ^+, we get the same inequality for $x < 0$. Hence $|k(x)|\, p(x) \leq M r(x)$, where $r(x) = \exp\left(q(x) - \alpha q\left(\frac{x}{\alpha}\right)\right)$. $q(x)$ being concave, we have $r(x) \leq 1$.

Let us first suppose that it is possible to choose an $\alpha > 1$ such that $r(x) \in L^1(-\infty, \infty)$. Then the above $k(x)$, constructed with this α, belongs to L_p^1, and its Fourier transform $K(it)$ is $\neq 0$ for all t. (In the special example, $r(x) \in L^1(-\infty, \infty)$ for every $\alpha > 1$.)

We shall now prove that $\varphi(x)$ and $k(x)$ satisfy the equation (25). We have

$$(46)\qquad \int_0^\infty k(x+y)\,\varphi(x)\,dx = \frac{1}{2\pi i}\int_0^\infty \varphi(x)\,dx \int_{\Gamma^-} K(s)\,e^{s(x+y)}\,ds =$$

$$= \frac{1}{2\pi i}\int_{\Gamma^-} K(s)\,e^{sy}\,ds \int_0^\infty \varphi(x)\,e^{sx}\,dx = \frac{1}{2\pi i}\int_{\Gamma^-} K(s)\,F(s)\,e^{sy}\,ds\,,$$

where we have used (45). The change in the order of integration is justified by absolute convergence. In the same way we get

$$(47)\qquad -\int_{-\infty}^0 k(x+y)\,\varphi(x)\,dx = \frac{1}{2\pi i}\int_{\Gamma^+} K(s)\,F(s)\,e^{sy}\,ds\,.$$

Remembering the representation (42) of $F(s)$ in D, we see that the right members in (46) and (47) are equal, and subtraction then gives (25).

If $r(x)$ does not $\in L^1(-\infty,\infty)$ for any $\alpha > 1$, the construction will be somewhat more complicated. We can set, for instance,

$$K(s) = \frac{1}{(a^2-s^2)^2}\int_s^{+i\infty} \frac{s-\omega}{G_0(\omega)\,(a^2-\omega^2)^2}\,d\omega\,.$$

It is clear that $K(it) \neq 0$ for all t, and, by the same argument as before, together with two partial integrations, we can prove that $k(x) \in L_p^1$. We can then change the definition of $G(s)$ in a similar way, such that $|K(s)\,G(s)| < M/|a^2-s^2|$, and such that $|\varphi(x)|/p(x)$ will be bounded and belong to $L^1(-\infty,\infty)$. The change in the order of integration in (46) and (47) can then be justified as before, and $\varphi(x)$ and $k(x)$ will be a pair of functions with the desired properties.

CHAPTER 4

ON TRANSLATIONS OF SEQUENCES IN $l^\infty(0, \infty)$

19. Definitions. — Let $l^\infty(0, \infty)$ be the space consisting of all sequences of complex numbers $A = \{a_n\}_0^\infty$, with the norm

$$\| A \| = \underset{0 \leq n < \infty}{\text{l.u.b.}} | a_n | < \infty,$$

and $l^1(0, \infty)$ the space of sequences $B = \{b_n\}_0^\infty$, with the norm

$$\| B \| = \sum_{n=0}^{\infty} | b_n | < \infty.$$

Let T denote a translation to the left, and let $A \in l^\infty(0, \infty)$. We now consider the linear manifold spanned by the elements

$$T^k A = (a_k, a_{k+1}, a_{k+2}, \ldots), \quad k = 0, 1, 2, \ldots$$

Let C_A be the weak closure of this manifold, i.e. an element $X = \{x_n\}_0^\infty \in C_A$ if and only if the equations

$$c_k = \sum_{n=0}^{\infty} a_{n+k} b_n = 0, \quad k = 0, 1, 2, \ldots, \tag{48}$$

where $B \in l^1(0, \infty)$, imply that

$$\sum_{n=0}^{\infty} x_n b_n = 0.$$

The spectral set of A, Λ_A, is defined as the set of all complex numbers λ, such that $\{\lambda^n\}_0^\infty \in C_A$. Λ_A is a closed set included in the closed unit circle $|\lambda| \leq 1$. If

$$\left(1, \binom{\nu+1}{\nu}\lambda, \binom{\nu+2}{\nu}\lambda^2, \ldots\right) \in C_A$$

for $\nu = 0, 1, \ldots, q-1$, but does not $\in C_A$ for $\nu = q$, we shall say that λ is a point of Λ_A of the multiplicity q. If $|\lambda| = 1$, the multiplicity must obviously be $= 1$. — If C_A is a proper subset of $l^\infty(0, \infty)$, we shall say, as before, that A is mean-periodic.

20. The extinction theorem in the weak topology.

Theorem 9. *If $A \in l^\infty(0, \infty)$ and is different from the null element, then Λ_A is not empty.*

This theorem is easily proved, using the theory of normed rings. It can also be proved by a method quite similar to that used in the proof of Theorem 1. This proof seems to be of more interest, as it also gives us some information about the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1, \tag{49}$$

associated to A, in the case when A is mean-periodic.

Let us suppose that A is mean-periodic. The space $l^1(0, \infty)$ will then contain an element B, different from the null element, such that the numbers c_k defined by (48) are $= 0$ for $k = 0, 1, 2, \ldots$ (For $k < 0$, c_k is still defined by (48), where we define $a_n = 0$ for $n < 0$.) Let us now set

$$g(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n},$$

$$h(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{z^n},$$

these functions being holomorphic for $|z| > 1$.

Using the same arguments as in sections 9 and 10 (where integrals now will be replaced by sums) it is an easy matter to prove the following theorem:

Theorem 10. *Let A be mean-periodic, and let $f(z)$ be defined by* (49). *Then $f(z)$ is a single-valued analytic function, regular in the whole plane, with the possible exception of singularities on $|z|=1$, forming a set of measure zero, and poles in $|z|>1$. For $|z|>1$, $f(z)$ can be represented in the form $h(z)/g(z)$ (where $g(z)$ and $h(z)$ are defined as above). The set of singularities of $f\left(\frac{1}{z}\right)$ is identical with Λ_A; to a pole λ in $|z|<1$ corresponds a point λ in Λ_A of the same multiplicity.*

21. The extinction theorem in the narrow topology. — Let $X_k=(x_{k,0}, x_{k,1}, x_{k,2}, \ldots)$, $k=1, 2, 3, \ldots$, be a sequence of elements in $l^\infty(0,\infty)$. This sequence is said to converge narrowly (see page 7) to an element $X=(x_0, x_1, x_2, \ldots)$ as $k\to\infty$ if

$$\lim_{k\to\infty} x_{k,n}=x_n \text{ for every fixed } n\geq 0,$$

and

$$\lim_{k\to\infty} \|X_k\| = \|X\|.$$

Let $A\in l^\infty(0,\infty)$. Using this narrow topology, we can define the set $\tilde{C}_A$ and the spectral set $\tilde{\Lambda}_A$ in the same manner as C_A and Λ_A was defined by aid of the weak topology in section 19.

Our principal aim in this chapter is to give an elementary proof of Theorem 9, with the weak topology replaced by the stronger narrow topology.

Theorem 11. *If $A\in l^\infty(0,\infty)$ and is different from the null element, then $\tilde{\Lambda}_A$ is not empty.*

Proof: Let us first suppose that $\tilde{C}_A$ contains an element $X=\{x_n\}_0^\infty$ such that

$$\lim_{n\to\infty} \|T^n X\| > 0. \tag{50}$$

Let $l^\infty(-\infty, \infty)$ denote the space, which consists of all sequences $D = \{d_n\}_{-\infty}^{\infty}$ with $\| D \| = \underset{-\infty<n<\infty}{\text{l.u.b.}} | d_n | < \infty$. To the above element X we associate the element

$$X' = (\ldots, 0, 0, x_0, x_1, \ldots)$$

in $l^\infty(-\infty, \infty)$. T denotes a translation to the left, as before,

$$TX' = (\ldots, 0, x_0, x_1, x_2, \ldots).$$

We now concider the sequence $T^n X'$, $n = 0, 1, \ldots$ in $l^\infty(-\infty, \infty)$. By the diagonal process, we can extract from this sequence a subsequence, which converges narrowly in $l^\infty(-\infty, \infty)$ to an element Y different from the null element. This follows from (50) and the fact that T is isometric. Now, if T^{-1} denotes a translation to the right in $l^\infty(-\infty, \infty)$ it is easy to see that every element $T^n Y$, $n = 0, \pm 1, \ldots$, is an element of $\tilde{C}_A$ (if we only consider components with indices ≥ 0).

Y being different from the null element, the narrow closure in $l^\infty(-\infty, \infty)$ of the linear manifold spanned by the set $T^n Y$, $-\infty < n < \infty$, must contain at least one element of the form $\{e^{in\varphi}\}_{-\infty}^{\infty}$. (See Beurling [2]; the proof of the extinction theorem there can be applied, almost without change, to prove this assertion. The above device is due to the same author, who has used it to prove the following theorem: Let $\varphi(x)$ be a function which is bounded and uniformly continuous on $(0, \infty)$ and is such that $\overline{\lim_{x\to\infty}} |\varphi(x)| > 0$. Then the narrow closure of the linear manifold spanned by the set $\varphi(x+\tau)$, $\tau \geq 0$, must contain a pure oscillation $e^{-i\lambda x}$.)

Consequently, when (50) is valid for one element in $\tilde{C}_A$, $\tilde{C}_A$ must contain an element $\{e^{in\varphi}\}_0^\infty$. Our interest will therefore now be concentrated on the case when the relation

$$\lim_{n\to\infty} \| T^n X \| = 0 \tag{51}$$

holds true for all $X \in \tilde{C}_A$. We shall prove that in this case the function $f(z)$ defined by (49) must be rational.

To this end, let us suppose that $f(z)$ is not rational. We shall show that this assumption leads to a contradiction.

Let E_0 denote the set of all elements in $\tilde{C}_A$ of the form

$$X = (1, x_1, x_2, \ldots), \quad \|X\| = 1.$$

E_0 is not empty. We now introduce the quantity

$$\alpha = \underset{X \in E_0}{\text{l.u.b.}} |x_1|.$$

We shall prove that $\alpha = 1$.

Let us suppose that $\alpha \neq 1$, i.e. $\alpha < 1$. E_0 must contain an extremal element

$$Y = (1, y_1, y_2, \ldots), \quad |y_1| = \alpha.$$

By (51), $|y_n| < \frac{1}{2}$ for all $n > N$. Furthermore, let us now suppose that $\tilde{C}_A$ contains an element where $N+1$ consecutive components are $0, 1, 0, \ldots, 0$. Then $\tilde{C}_A$ will contain an element of the form

$$(0, 1, 0, \ldots, 0, z_{N+1}, z_{N+2}, \ldots). \tag{52}$$

We now multiply this element by εy_1 (if $y_1 \neq 0$; otherwise by ε), and add the result to Y. In this way we obtain the element

$$Z = (1, (1+\varepsilon) y_1, y_2, \ldots, y_N, y_{N+1} + \varepsilon y_1 z_{N+1}, \ldots),$$

which belongs to $\tilde{C}_A$. Obviously, we can choose $\varepsilon > 0$ so small that $Z \in E_0$.

However, this is a contradiction, the second component in Z having a modulus $> \alpha$. Hence, if $\alpha < 1$, $\tilde{C}_A$ does not contain any element of the form (52). Let us then form the element

$$\xi_0 A + \xi_1 T A + \ldots + \xi_N T^N A \tag{53}$$

belonging to $\tilde{C}_A$. From what precedes it will be seen that the system of equations

$$\begin{cases} \xi_0 a_n + \xi_1 a_{n+1} + \ldots + \xi_N a_{n+N} = 0, \\ \xi_0 a_{n+1} + \xi_1 a_{n+2} + \ldots + \xi_N a_{n+N+1} = 1, \\ \vdots \\ \xi_0 a_{n+N} + \xi_1 a_{n+N+1} + \ldots + \xi_N a_{n+2N} = 0, \end{cases}$$

has no solution for any $n \geq 0$, i.e.

$$\begin{vmatrix} a_n & \ldots & a_{n+N} \\ \vdots & & \vdots \\ a_{n+N} & \ldots & a_{n+2N} \end{vmatrix} = 0$$

for all $n \geq 0$. Thus $f(z)$ is rational, and this contradiction shows that $\alpha = 1$.

Let then E_1 be the set of all elements in $\tilde{C}_A$ of the form

$$X = (1, x_1, x_2, \ldots), \quad |x_1| = 1, \quad \|X\| = 1.$$

We have just proved that E_1 is not empty, and, by the same argument as above, we can conclude that $\underset{X \in E_1}{\text{l.u.b.}} |x_2| = 1$. Repeating the argument, it will be seen that $\tilde{C}_A$ contains a sequence $\{A_n\}_0^\infty$ of elements, where A_n has the property that the first component is 1 and the n following components are of modulus 1, and $\|A_n\| = 1$. From this sequence we can extract a subsequence which converges narrowly to an element in $\tilde{C}_A$, having all its components of modulus 1. We thus get an element for which (51) is not valid, and from this contradiction it results that $f(z)$ is rational.

If $f(z)$ is a constant, $0 \in \Lambda_A$. If $f(z)$ is not a constant, then $f(z)$ must have a pole $1/\lambda$, $|\lambda| < 1$. As $f(z)$ is rational, it is possible to choose a finite number of constants $\xi_0, \xi_1, \ldots, \xi_N$ such that the element (53) reduces to $\{\lambda^n\}_0^\infty$. This completes the proof of Theorem 11.

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